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# The search for weak harmonic signals in a spectrum with application to gravity data

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**Abstract**

When considering the search for discovery or amplitude estimation of a spectral line with a probabilistic approach, great attention must be paid to the meaning of each step. We give the probability law for the amplitude of a spectral peak in the presence of random noise appearing in a periodogram and discuss the effective probability of the existence of the corresponding wave. We find that the estimated amplitude of a spectral peak is biased and should be corrected when the signal-to-noise ratio is small. As a first application to gravity data, it results in a re-estimation of the gravimetric amplitude factors (delta factors) provided by least-squares tidal analysis. We also estimate the probability of observing a spectral line above a given level in the spectrum of a purely random noise. This allows us to compute for given spectrum the number of peaks expected to overcross the classical levels used in statistical analysis (like $n\sigma$, where $\sigma$ is the standard deviation of the temporal noise distribution and $n$ is an integer with typical values equal to 2 or 3). A specific application to real data is investigating the gravity spectrum derived from a 5 year record of the French superconducting gravimeter and we show that the predicted statistics are indeed in agreement with the observations. We also show the statistical consequence of using longer observing periods to obtain the spectral estimations. The problem of detecting translational motion of the Earth's solid inner core (Slichter modes) in a gravity spectrum is analyzed and the probabilities of having a triplet of random peaks thresholding specific levels in a given frequency window are computed. We show that, in the case of a typical gravity spectrum (1 year of hourly data and a frequency window of 0.03 cycle $h^{-1}$), the probability of having a random set of three peaks exceeding a level of 3 $\sigma$, is very high. This emphasizes the need for a very careful analysis of spectral lines before inferring the existence of a true physical signal.

## 1. Introduction

The hunt for new seismic, gravimetric or rotational modes is a current activity among geophysicists. Most of the time, spectral analysis is used to investigate the different corresponding signals. Several techniques are involved: the correlogram method which provides power spectral density estimates (based on the Wiener–Kinchine theorem), the periodogram method which is equivalent to a least-squares approach and parametric methods like MEM (maximum entropy method) or ARMA (autoregressive moving average

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method) (see, e.g. Marple, 1987). Hereafter we will only consider the periodogram approach.

In a spectral peak detection problem, two major cases may arise:

(1) the peak amplitude is much larger than the surrounding noise and nobody doubts the existence of such a signal;

(2) the peak hardly exceeds the noise level. It is frequent to observe a lot of peaks slightly emerging from the noise. How do we select those which are significant?

To answer this question, one must adopt a probabilistic point of view. It means that the observed signal is considered as a result of a random experiment. The first question may be: if a peak does exist, what is the probability density describing the measured amplitude (with respect to the effective one)? A second question is: when detecting a peak, can we estimate any probability of its real existence?

In the first part of this paper, we consider the probability law of a sinusoidal wave amplitude in a spectrum when random white noise is present. It relates to the amplitude observed on a spectrum or, in an equivalent way, to the amplitude obtained by a least-squares computation. In the second part, we discuss the problem of the detection of a spectral peak with special attention paid to the search for Slichter modes (translational modes of vibration of the Earth's inner core) in gravity and propose an approach using the decision theory.

## 2. The probability law of a peak in the presence of noise

Periodograms are derived by application to the signal of the Discrete Fourier Transform (DFT) defined by

$$X_n = \frac{1}{N}\sum_m x_m \exp\left(\frac{-i2\pi mn}{N}\right)$$
$$x_m = \sum_n X_n \exp\left(\frac{i2\pi mn}{N}\right) \tag{1}$$

These formulae between $N$-point time sequence $x_m$ and $N$-point transform sequence $X_n$ are strictly equivalent. However, if we determine $X_n$ by using a least-squares method, i.e. by minimizing

$$\sum_m \left(x_m - \sum_n X_n \exp\frac{i2\pi mn}{N}\right)^2$$

we obtain the same value for $X_n$. The DFT is hence equivalent to the least-squares method.

Parseval's theorem can be written

$$\sum_n |X_n|^2 = \frac{1}{N}\sum_m |x_m|^2 \tag{2}$$

Hence, when only noise is present, we get

$$\sigma_X = \frac{\sigma_x}{\sqrt{N}} \tag{3}$$

where $\sigma_x$ and $\sigma_X$ are the standard deviations of the time signal and its spectrum, respectively. Consequently, the non-centered variance of the spectrum decreases like $1/N$.

On the other hand, when a periodic oscillation of given amplitude is present in the noise free case, the corresponding Fourier coefficient (see Eq. 1) remains equal to its amplitude (see, e.g. Florsch et al., 1991); this type of spectrum is often called a normalized Fourier amplitude spectrum. This is easily understandable by noting that a wave remains identical to itself when changing the observed time span while the random noise is never the same. Increasing the observed time duration leads indeed to a reduction of the noise by a factor $1/\sqrt{N}$, where $N$ is the number of time samples.

Consider now the mixed case where some noise is present in addition to a periodic signal and study the resulting influence in estimating the spectral wave amplitude. We denote $A$ the amplitude of an existing wave at a given frequency, $A_R$ and $A_I$ its real and imaginary part, respectively. Let the noise be $X = X_R + iX_I$ at the same frequency (estimated from the surrounding noise mean level on the spectrum).

Since the DFT is linear, we have for the total spectral signal $S_R + S_I$ (at the same frequency)

$$S_R = A_R + X_R$$
$$S_I = A_I + X_I$$

The amplitude spectrum (at this frequency) is given by

$$S = \sqrt{S_R^2 + S_I^2} = \sqrt{(A_R + X_R)^2 + (A_I + X_I)^2} \quad (4)$$

while its phase $\phi$ is

$$\tan \phi = \frac{A_I + X_I}{A_R + X_R}$$

In this expression, $X_R$ and $X_I$ must be considered as random variables. Hence, $S$ and $\Phi$ are also random variables. Even if $x_n$ is not Gaussian, $X_R$ and $X_I$ tend to be Gaussian since they result from a linear combination of numerous terms (Central limit theorem). Therefore we consider in the following that $X_R$ and $X_I$ are independent Gaussian random variables having the following probability law

$$P_{X_R}(X_R) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{X_R^2}{2\sigma^2}\right) \quad (5)$$

and similarly for $P_{X_I}(X_I)$ by exchanging $X_R$ and $X_I$ ($\sigma$ is the standard deviation of the noise).

To find the probability law of $S$, one has to use the transformation

$$(X_R, X_I) \mapsto \sqrt{(A_R + X_R)^2 + (A_I + X_I)^2} = S(X_R, X_I) = S$$

$$(X_R, X_I) \mapsto \tan^{-1}\left(\frac{A_I + X_I}{A_R + X_R}\right) = \phi(X_R, X_I) = \phi$$

Inverting these relations yields

$$X_R = S \cos \phi - A_R = X_R(S, \phi)$$
$$X_I = S \sin \phi - A_I = X_I(S, \phi) \quad (6)$$

Denote $P_{S,\phi}$ the probability law for the couple of variables $(S, \phi)$ and $P_{X_R,X_I}$ the one for $(X_R, X_I)$.

We use the following transformation valid for random variables

$$P_{S,\phi}(S, \phi) = P_{X_R,X_I}[X_R(S, \phi), X_I(S, \phi)] J \quad (7)$$

where $J$ is the jacobian of the functions $X_R$ and $X_I$

$$J = \left| \frac{\partial X_R}{\partial S} \frac{\partial X_I}{\partial \phi} - \frac{\partial X_I}{\partial S} \frac{\partial X_R}{\partial \phi} \right| = S$$

The probability law for the noise is

$$P_{X_R,X_I}(X_R, X_I) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{X_R^2 + X_I^2}{2\sigma^2}\right) \quad (8)$$

As said before, $X_R$ and $X_I$ are supposed to be independent and of variance $\sigma^2$. Hence

$$P_{S,\phi}(S, \phi) = \frac{S}{2\pi\sigma^2} \exp\left\{\frac{-1}{2\sigma^2}\left[(S \cos \phi - A_R)^2 + (S \sin \phi - A_I)^2\right]\right\} \quad (9)$$

To obtain the probability law for the amplitude $S$ alone, one has to compute the marginal probability

$$P_S(S) = \int_0^{2\pi} P_{S,\phi}(S, \phi)\, \mathrm{d}\phi \quad (10)$$

and, similarly, for the phase

$$P_\phi(\phi) = \int_0^{\infty} P_{S,\phi}(S, \phi)\, \mathrm{d}S \quad (11)$$

After performing these calculations, we get the following results

$$P_S(S) = K_1 \exp\left(-\frac{S^2}{2\sigma^2}\right) I_0(SK_2)$$

$$P_\phi(\phi) = \frac{1}{2\pi}\left(K_1\sigma^2 + \sqrt{\frac{\pi}{2}} \exp\left[-\frac{K_1 \sin^2(\theta - \phi)}{2\sigma^2}\right] \times \left\{1 + \operatorname{erf}\left[\frac{A \cos(\theta - \phi)}{\sqrt{2}\,\sigma}\right]\right\}\right) \quad (12)$$

with

$$K_1 = \frac{\exp\left(-\frac{A^2}{2\sigma^2}\right)}{\sigma^2},$$

$$K_2 = \frac{A}{\sigma^2}$$

and the error function erf being defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{2}}\int_0^x \exp(-u^2)\ \mathrm{d}u$$

$I_0$ being the modified Bessel function of order 0 and $\theta$ being the phase of the noise.

### 2.1. Analysis of the amplitude response

The law $P_S(S)$ is called by radarists the 'Rice–Nakagami' distribution. In Fig. 1 we can see the shape of this law for a deterministic wave of amplitude 1 and for different values of $\sigma$. It is important to notice that the law is not symmetric, especially for low values of $A/\sigma$. It results in a bias when evaluating the amplitude using a least-squares approach and this point is discussed later in more detail.

For small values of $A/\sigma$, we obtain the limit

$$P_S(S) \sim \frac{S}{\sigma^2}\exp\left(-\frac{S^2}{2\sigma^2}\right) \tag{13}$$

This is the well-known Rayleigh law for pure noise. The mean of the Rayleigh law is $\sigma\sqrt{\pi/2}$, its variance $2\sigma^2$ and its centered variance $(2-\pi/2)\sigma^2$.

For large values of $A/\sigma$ we have to the first order and using the asymptotic behavior of the modified Bessel function $I_0(x) \sim e^x/\sqrt{(2\pi x)}$

$$P_S(S) \sim \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(A-S)^2}{2\sigma^2}\right] \tag{14}$$

The law tends to become Gaussian, centered on $A$ with a standard deviation equal to the parameter $\sigma$ of the Rayleigh law.

### 2.2. Consequences of the dispersion

We illustrate this point by performing a numerical experiment shown in Fig. 2. Several waves (19) of constant amplitudes are superimposed on to random white noise with a (Rayleigh law) parameter $\sigma = 0.4$. Fig. 2(a) shows the spectrum

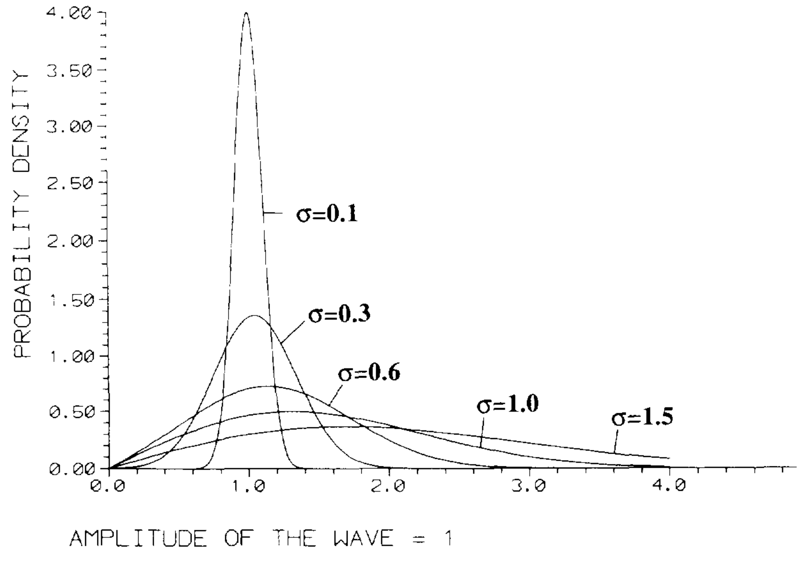


Fig. 1. Rice–Nakagami distribution. It gives the law of probability of a spectral line in the presence of noise. The amplitude of the 'wave' is 1 and the parameter of the Rayleigh law describing the noise is $\sigma$. The location of the maximum of the distribution shifts to the right (larger estimated amplitude) as a function of the value of $\sigma$ and is responsible for a bias in spectral analysis.

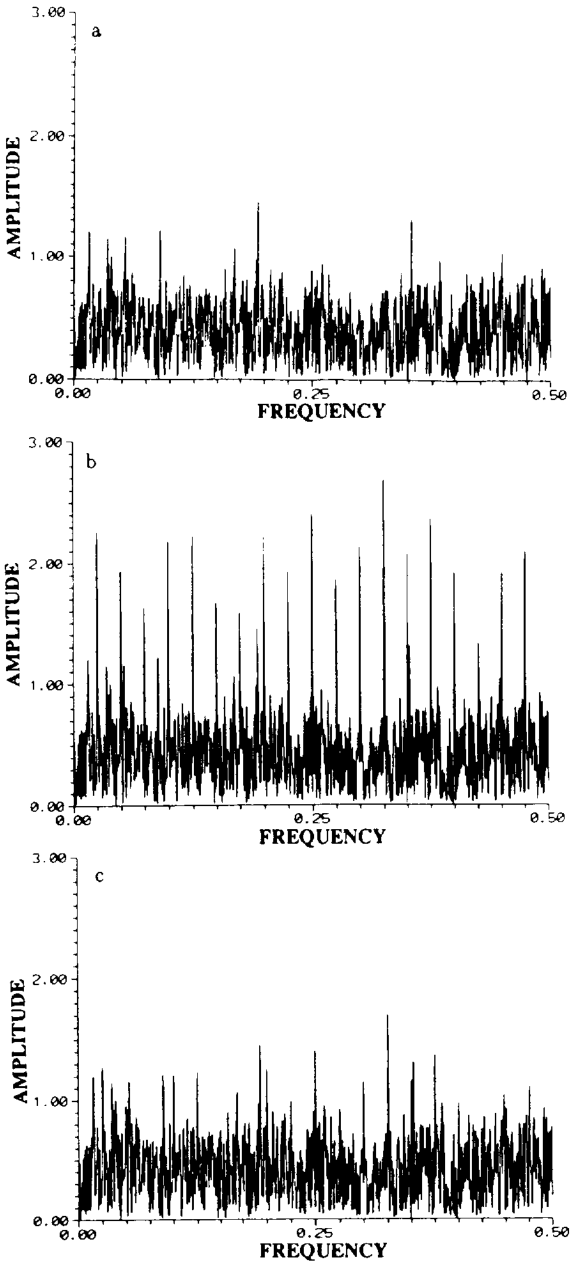


Fig. 2. Numerical experiment showing the amplitude spectra of 19 waves in the presence of random white noise. The parameter of the Rayleigh law describing the noise distribution is $\sigma = 0.4$. (a) Shows the spectrum of the noise alone; in (b) the amplitude of the waves is 2 at the frequencies 0.025 $k$ ($k = 1, \ldots 19$) and one can notice the dispersion resulting from the presence of the noise; in (c), the amplitude of the waves is 1 (at the same frequencies as in (b)) and it becomes hard to distinguish signal from noise.

of the noise alone. On Fig. 2(b), the amplitude of the waves equals 2. The noise clearly causes a change in the amplitude of the waves. In Fig. 2(c), the amplitude is 1 and it becomes difficult to distinguish the signal from the noise. This shows the difficulty in attempting to characterize any amplitude variation of a signal by taking successive time windows. A typical example is the search in Earth tides for time variations in the tidal gravimetric amplitude factors ($\delta$ factors) through successive tidal analyses. The classical procedure in (gravity) tidal analysis consists of estimating from gravity data the amplitude and phase of a number of tidal waves and comparing them with theoretical tides (as would be observed on a rigid Earth) (see, e.g. Melchior, 1983; Dehant and Ducarme, 1987). This leads then to the so-called gravimetric amplitude ($\delta$) and phase ($\kappa$) factors which express the ability of the Earth to (an)-elastically deform when submitted to tidal forcing. As shown by Fig. 2, the noise content of the gravity signal affects the determination of the tidal gravimetric factors, especially for small-amplitude waves, and this point has to be considered before extracting the full geophysical information from the results of tidal analyses.

### 2.3. *Consequence of the asymmetry of the probability law*

We have seen that the probability law for the amplitude of a given wave in the presence of noise is not symmetric. This asymmetry results in a systematic bias if the amplitude is estimated using a least squares method (or equivalent). Indeed, it is well known that the least-squares solution equals the mean of the probability law representing the solution (see, e.g. Tarantola, 1987). Three parameters are involved: (1) the effective amplitude of the wave (say $A$); (2) the maximum of the probability law (say $S_{max}$); (3) the mean of the probability law (L.S. solution), say $S_{mean}$.

The maximum is provided by nulling its derivative. Setting $b = S/A$ and $e = (\sigma/A)^2$, one has to solve

$$I_0(b/e)(e/b - b) + I_1(b/e) = 0 \tag{15}$$

where $I_0$ and $I_1$ are the modified Bessel functions. This can be done by using the Newton–Raphson Method. The mean of the law is simply

$$S_{mean} = K_1 \int_0^{\infty} S^2 \exp\left(-\frac{S^2}{2\sigma^2}\right) I_0(S/\sigma^2)\, dS \quad (16)$$

The Gauss integration scheme is useful for this computation. Fig. 3 shows, for a wave of amplitude $A = 1$, the two parameters $S_{max}$ and $S_{mean}$. One can see, for instance, that the L.S. amplitude is 4% overestimated when the ratio $\sigma/A = 0.3$. At first glance, this does not seem dramatic since we have $1.04 \pm 0.3$. In fact, when considering a larger number of waves, this bias becomes really problematic. If a hundred waves are involved, the random error is reduced by a factor 10 while the systematic bias remains and we have then $1.04 \pm 0.03$ leading to a wrong estimate.

A similar case arises for instance in tidal analysis. Basically, in the estimate of the value of the gravimetric amplitude factors ($\delta$ factors) we are facing the problem discussed above dealing with a periodic signal in the presence of noise (instrumental or natural). The accurate knowledge of the tidal gravimetric factors is essential for various studies like the retrieval of the Free Core Nutation (FCN) resonance parameters (e.g. Neuberg et al., 1987; Florsch et al., 1994). Notice that the bias has almost no importance in the search for the latitude dependence induced by the Earth's rotation and ellipticity (Wang, 1994) because of the large ratio signal/noise of the waves $O_1$ and $M_2$ which are involved. More generally, any spectral consideration involving a stacking or averaging method applied to data sets from different stations or relative to different time spans (e.g. Smylie et al., 1993a,b) is relevant to this problem.

A simple example showing the influence of the asymmetry of the probability law is provided by averaging $\delta$ factors from two different analyses; notice that this process of averaging $\delta$ factors for different waves is merely for the purpose of demonstration and has no geophysical meaning in itself. Table 1 shows these factors which are relative to small tidal waves (the large waves like $K_1$, $O_1$, etc. are not considered here because of their extremely small ratio $\sigma/A$ leading to a negligible bias); both analyses are relative to the same time

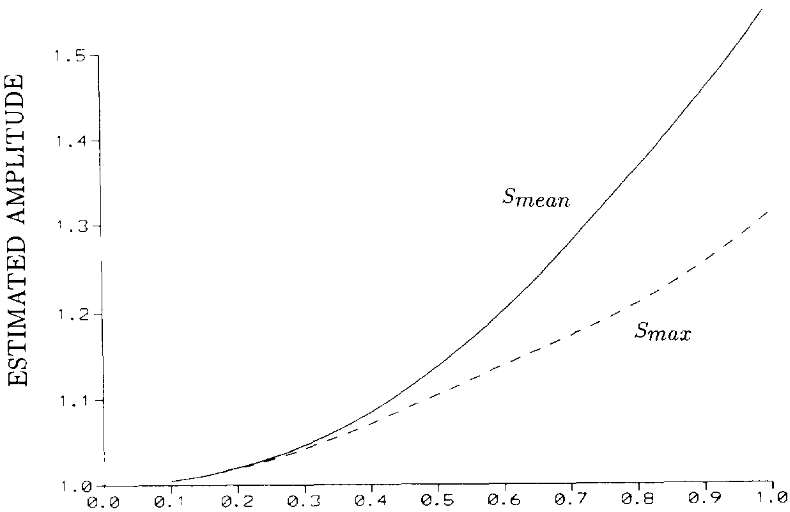


Fig. 3. Bias as a function of noise level. For a wave of amplitude 1, it shows the estimated spectral amplitude according to the noise (Rayleigh) parameter $\sigma$. The continuous line corresponds to the least-squares solution (i.e. the amplitude found by spectral analysis) while the dashed line represents the location of the maximum of the law. This curve allows one to correct the estimated amplitude for the bias caused by noise.

Table 1
Tidal gravimetric factors relative to two different least-squares analyses relative to the same time period

| Tidal wave | 'Good' analysis | | 'Bad' analysis | |
|---|---|---|---|---|
| | $\delta$ | r.m.s. | $\delta$ | r.m.s. |
| $\sigma Q_1$ | 1.1521 | 0.0110 | 1.130 | 0.14 |
| $2Q_1$ | 1.1506 | 0.0035 | 1.077 | 0.047 |
| $\sigma_1$ | 1.1490 | 0.0030 | 1.200 | 0.039 |
| $\tau_1$ | 1.1654 | 0.0094 | 1.332 | 0.15 |
| $MM_1$ | 1.1434 | 0.0119 | 1.187 | 0.15 |
| $M_1$ | 1.1508 | 0.0011 | 1.108 | 0.056 |
| $\chi_1$ | 1.1455 | 0.0060 | 1.075 | 0.076 |
| $S_1$ | 1.1933 | 0.0143 | 1.366 | 0.187 |
| $\psi_1$ | 1.2589 | 0.0093 | 1.308 | 0.125 |
| $\phi_1$ | 1.1698 | 0.0054 | 1.062 | 0.080 |
| $\theta_1$ | 1.1546 | 0.0062 | 1.137 | 0.075 |
| $SO_1$ | 1.1596 | 0.0068 | 1.283 | 0.088 |
| $\epsilon_1$ | 1.1735 | 0.0074 | 1.200 | 0.085 |
| $\epsilon_2$ | 1.1521 | 0.0117 | 1.096 | 0.134 |
| $2N_2$ | 1.1540 | 0.0036 | 1.161 | 0.046 |
| $\alpha_2$ | 1.2028 | 0.0219 | 1.2216 | 0.207 |
| $\beta_2$ | 1.2584 | 0.0327 | 1.724 | 0.306 |
| $\lambda_2$ | 1.1913 | 0.0141 | 1.447 | 0.148 |
| $L_2$ | 1.2111 | 0.0045 | 0.165 | 0.029 |
| $T_2$ | 1.1892 | 0.0037 | 0.170 | 0.041 |
| $R_2$ | 1.1905 | 0.0211 | 1.051 | 0.221 |
| $\chi_2$ | 1.1684 | 0.0557 | 1.818 | 0.479 |
| $\eta_2$ | 1.1850 | 0.0088 | 1.110 | 0.095 |
| $M_3$ | 1.0661 | 0.0035 | 1.036 | 0.017 |

The selected waves are only small waves for which the small signal-to-noise ratio leads to some bias in the determination of the factors. The 'good' analysis refers to data that have been cleaned for large spikes and offsets before tidal fitting. The 'bad' analysis refers to raw uncorrected data before tidal fitting.

signal. In the 'bad' case, glitches and spikes were not removed and the tidal fit was done on raw gravity. On the contrary, the signal was cleaned before the second analysis ('good' case) therefore reducing the noise contribution (Florsch et al., 1991; Hinderer et al., 1994). We see that only 12 waves (of 24) have greater amplitudes in the 'bad' analysis; however, in order to point out the trend, averaging the $\delta$ and taking into account individual errors leads to $\delta_{mean} = 1.228 \pm 0.16$ in the 'bad' case and to $\delta_{mean} = 1.172 \pm 0.016$ in the 'good' case. The value in the more noisy 'bad' case is larger than the other one as expected, but this test is not fully significant since the error bars are compatible with the bias. What we can expect is that it would no longer be true if the number of the averaged waves was higher (reducing the error bars but not the bias); another way to show the bias effect would be to average for each wave the tidal factors of successive partial analyses of similar length.

Fig. 3 provides a means to correct the bias. One possible immediate application concerns the FCN frequency and damping retrieval from the corrected amplitude of diurnal tidal waves of frequency close to the FCN eigenfrequency (see, e.g. Neuberg et al., 1987; Florsch et al., 1994).

## 3. Significance of a peak detection

In physical phenomena where periodic signals are to be expected (e.g. tides, seismic normal modes, gravity-inertial oscillations of the fluid outer core, translation of the inner core), the most useful approach is the spectral one. We tackle now the following problem: when considering a specific peak in a spectrum, to what extent can we state that it is really significant? Before answering this question, one has to understand carefully how the noise interferes with the signal and may cause spurious peaks.

Consider first a pure random (assumed to be Gaussian) signal. We have seen that the spectrum of this signal is also random and follows the Rayleigh law with parameter $\sigma$

$$R(S) = \frac{S}{\sigma^2}\exp\left(-\frac{S^2}{2\sigma^2}\right) \quad (17)$$

If the time signal contains $2N$ points, the spectrum contains $N$ degrees of freedom since only positive frequencies are included (from now on, $N$ refers to degree of freedom and not to the number of time samples).

Consider the spectrum at a specific frequency. The probability of having an amplitude exceeding the threshold $k\sigma$ is given by

$$P(S > k\sigma) = \int_{k\sigma}^{\infty} R(S)\ \mathrm{d}S \quad (18)$$

When dealing with the whole spectrum, we consider $N$ independent realizations of the same random experiment. If we note $p = P(S > k\sigma)$ the probability for one peak to exceed $k\sigma$, one can estimate the probability of seeing $m$ peaks exceeding this threshold over the whole spectrum. In such an experiment, the binomial law applies. We have then

$$P(m \text{ peaks} > k\sigma) = P_m(k) = \binom{N}{m} p^m (1-p)^{N-m} \qquad (19)$$

with

$$\binom{N}{m} = \frac{N!}{m!(N-m)!}$$

Hence we get the probability of having no peak above $k\sigma$

$$P(0 \text{ peak} > k\sigma) = P_0(k) = \binom{N}{0} p^0 (1-p)^N = (1-p)^N \qquad (20)$$

This result is obvious when considering the event 'the peak does not exceed the threshold' which is repeated $N$ times.

Suppose that we have a spectrum with $N =$ 1024 points. At the level of $2\sigma$, the probability of having the peak exceeding this threshold is $p =$ 0.135355. Since the binomial law applies, the number of waves above this level is on average $pN = 139$ peaks! This simple example gives some insight into the meaning of any peak detection in a spectrum. We obtain the following probabilities of seeing no peak at the levels $2\sigma$, $3\sigma$, $4\sigma$ and $5\sigma$ ($p$ being the probability at one frequency to exceed the corresponding threshold)

$$\begin{aligned} p &= 0.135335 \rightarrow P_0(2) = 2.1 \times 10^{-65} \\ p &= 0.0111090 \rightarrow P_0(3) = 1.1 \times 10^{-5} \\ p &= 0.00033546 \rightarrow P_0(4) = 0.71 \\ p &= 0.0000037267 \rightarrow P_0(5) = 0.9962 \end{aligned} \qquad (21)$$

The opposite event, 'there is at least one peak above $k\sigma$', has the probability $1 - P_0$. For instance, at the $3\sigma$ level, we have a $1 - 1.1 \times 10^{-5}$ $= 0.999989$ chance of seeing at least one peak!

These considerations show how careful one should be before claiming the detection of a

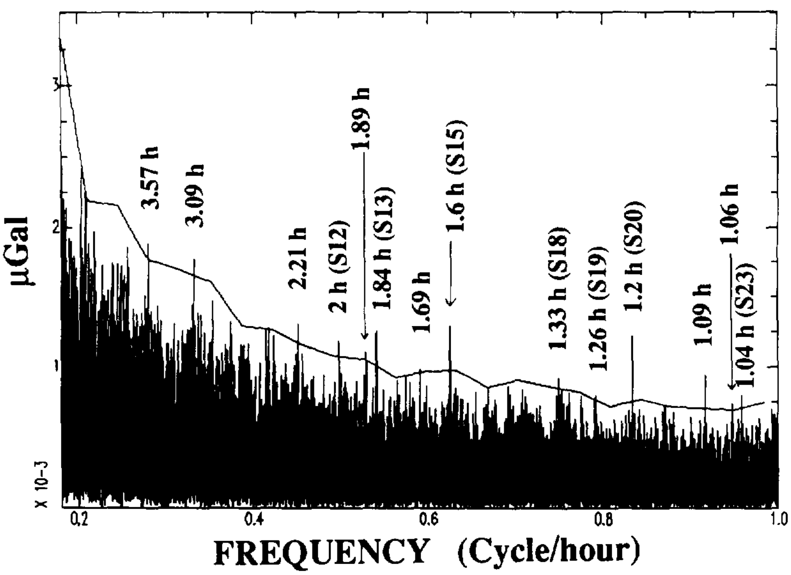


Fig. 4. Spectrum of the gravity signal from the superconducting gravimeter located in Strasbourg between 1 and 5 h periods. The threshold $4\sigma$ is overpassed by several spectral lines. Some of them are harmonics of the solar day (denoted $S_n$, where $n$ is the number of the overtone) and correspond to physical signals. The other peaks are probably only random excursions of the noise and their number (seven) is in fair agreement with the number expected from statistics (see text). The vertical scale is in $\mu$gal $\times 10^{-3}$.

spectral peak; basically, below a high confidence level, it is virtually impossible to distinguish noise from signal.

Figure 4 is the amplitude spectrum of real data. It concerns 520 000 samples (1 sample per 5 min) of the gravity record from the French superconducting gravimeter installed near Strasbourg. Consequently, the spectrum involves 260 000 points. It is shown between 0.18 (5.6 h period) and 1 cycle $h^{-1}$. The Nyquist frequency is 6 cycle $h^{-1}$. The degree of freedom of the whole spectrum is $N = 260\,000$ but the degree of freedom in a specific window should be replaced by the number of independent spectral points in this window which is simply

$$\frac{f_2 - f_1}{\Delta f},$$

where $[f_1, f_2]$ is the frequency range of the window ($\Delta f = 1/(n_t \Delta t)$ is the fundamental frequency resolution depending on the number of time samples $n_t$ equally spaced by $\Delta t$). In this case, the practical degree of freedom of the spectrum in Fig. 4 is 520 000 (1 – 0.18)/12 (the division by 12 comes from the fact that the frequencies $f_1$ and $f_2$ are expressed in cycle per hour and the sampling interval $\Delta t$ is 5 min or 1/12 h); this value must be further divided by a factor 2 since a Hanning taper has been applied to the time sequence, leading to a degree of freedom value close to 17 800.

In Fig. 4, it clearly appears that the parameter $\sigma$ of the Rayleigh law describing the noise is not constant but depends on the frequency. We have computed it just by filtering the spectrum — providing the mean — and by applying the right coefficient to fit $4\sigma$ with respect to the theoretical mean of the Rayleigh law. For instance, let the threshold be $4\sigma$. The probability at a given frequency to overpass this value is $p = 0.00033546$. Hence, the probability of having no peak exceeding $4\sigma$ is $(1-p)^N = (1 - 0.00033546)^{17800} = 0.00255$. The binomial law indicates that the mean number of peaks exceeding this threshold is

$$pN \pm \sqrt{Np(1-p)} = 6 \pm 2.5$$

In Fig. 4, some of the spectral peaks which stick out clearly from the ambient noise are high-order harmonics of the solar day ($S_1$) and may correspond to real physical (thermally induced) signals at our station. Other peaks above the previous threshold have probably only a random meaning. By the way, it is interesting to note that the statistical results were predicting $6 \pm 2.5$ peaks and, from Fig. 4, we found indeed 7 peaks.

### 3.1. Influence of doubling the observation duration

This case arises when considering the signal from two independent time periods. What happens when doubling the number of data? We first emphasize that it is not at all obvious that signals provided by two remote instruments during the same period can be regarded as independent. Indeed, in the case of gravity records, the dominant observed noise is not instrumental but rather of natural origin (air pressure, ground vibrations). For a single station, the noise can be seen as random. However there are gravity variations which are induced by world-wide phenomena like atmospheric/oceanic loads and this questions the validity of the independence of the noise content between two remote stations; clearly, among other reasons, it will be a function of the distance separating the stations. On the other hand, the independence assumption is certainly more valid when observing two independent time intervals. Different schemes are possible:

(1) one performs a spectrum relative to each signal and deals with the joint random experiment;

(2) one takes the mean of the two individual spectra;

(3) the two signals may be linked and one performs the spectrum of the whole signal.

#### 3.1.1. Case (1)

The two spectra are regarded as the result of two independent random experiments. If $p$ is the probability of exceeding a given level in one spectrum, $p^2$ is the probability of exceeding the same level in both spectra. Hence, the probability of having no peak exceeding $k\sigma$ in both spectra is

$$P_0(k) = (1 - p^2)N \qquad (22)$$

where $N$ is the number of independent points in one spectrum.
We obtain (with $N = 1024$)

$$P_0(2) = 6 \times 10^{-9}$$
$$P_0(3) = 0.88 \tag{23}$$
$$P_0(4) = 0.99988$$

In comparison with the previous case where half of the duration was considered, we see that the situation has completely changed for the threshold $3\sigma$ (see Eqs. 21 and 23). This demonstrates that the duration should be in any case as long as possible to increase the probabilities $P_0(k)$ but, depending on the specific threshold (e.g. $4\sigma$), the qualitative difference is not very important (at least for this value of $N$).

*3.1.2. Case (2)*

The spectra are averaged. We have to find the law of probability for the sum of two random variables, each one following the Rayleigh law. It is given by the auto-convolution of the Rayleigh law. For a normalized parameter $\sigma = 1$, the Rayleigh law is

$$R(S) = S \exp\left(-\frac{S^2}{2}\right) \tag{24}$$

Hence the law for the sum is given by

$$\begin{aligned} R_2(S) &= \int_0^\infty x \exp(-x^2/2)(S-x) \\ &\quad \times \exp\left[-(S-x)^2/2\right] \mathrm{d}x \\ &= \frac{S}{2}\exp(-S^2/2) + \mathrm{erf}(S/2) \\ &\quad \times \exp(-S^2/4)\sqrt{\pi}/2(s^2/2-1) \end{aligned} \tag{25}$$

To obtain the law for the mean, we transform the scale as follows

$$\begin{aligned} R_{2_{\mathrm{mean}}} &= 2R_2(2S) = 2s \exp(-2S^2) + \mathrm{erf}(S) \\ &\quad \times \exp(-S^2)\sqrt{\pi}(2s^2-1) \end{aligned} \tag{26}$$

and we obtain the following probabilities

$$P(S > 2\sigma) = 0.064858\ldots = p_2$$
$$P(S > 3\sigma) = 0.00065621\ldots = p_3 \tag{27}$$
$$P(S > 4\sigma) = 0.7978536 \times 10^{-6} = p_4$$

Hence the probability of having no peak exceeding $2\sigma$, $3\sigma$ or $4\sigma$ is

$$P_0(2) = (1-p_2)^N = 0.135 \times 10^{-29}$$
$$P_0(3) = (1-p_3)^N = 0.5106 \tag{28}$$
$$P_0(4) = (1-p_4)^N = 0.99918$$

*3.1.3. Case (3)*

The signal is twice as long and the probability of having no peak is of the form $(1-p)^{2N}$. However, in this case, the parameter $\sigma$ of the Rayleigh law is changed. Multiplying the length of the signal by a factor 2 leads to division of $\sigma$ by a factor $\sqrt{2}$. To discuss the probability with respect to the initial spectrum of parameter $\sigma$, we must consider the probability of exceeding $\sqrt{2}k\sigma$ instead of $k\sigma$.
We have

$$P(S > 2\sqrt{2}\sigma) = p_2 = 0.01832$$
$$P(S > 3\sqrt{2}\sigma) = p_3 = 0.0001246 \tag{29}$$
$$P(S > 4\sqrt{2}\sigma) = p_4 = 1.127 \times 10^{-7}$$

Hence, the probability of having no peak exceeding $2\sigma$, $3\sigma$ or $4\sigma$ is

$$P_0(2) = 3.6 \times 10^{-17}$$
$$P_0(3) = 0.7747 \tag{30}$$
$$P_0(4) = 0.99978$$

The three cases (1, 2, 3) lead to different probability values since they are not strictly equivalent. However their probabilities relative to a given threshold are of the same order of magnitude. In conclusion, the advantage of choosing one or the other when the number of observations is doubled is not very important.

In some spectral analyses, a larger number of individual spectra are stacked together. It is the case for instance when using a moving window and averaging the spectrum obtained for each window (see, e.g. Smylie et al., 1993a,b). The detailed statistics corresponding to this more general case are not discussed here.

### 3.2. Application to the search for Slichter modes

The Slichter modes consist of three translational oscillations of the Earth's solid inner core. Different theories (e.g. Crossley et al., 1992; Smylie et al., 1992) have recently provided different frequency triplets for these modes. From the observational point of view, there have been numerous attempts to detect these modes. Very recently, there has been a claim of detection by Smylie (1992) using a stacked gravity spectrum from four European superconducting gravimeters (see also Smylie et al., 1993a,b). Unfortunately, the claimed triplet could not be confirmed in a stack of 2 year homogeneous records from the French and Canadian superconducting gravimeters, which were processed in an identical way (Jensen et al., 1992; Hinderer et al., 1994). However, this 2 year common time span did not overlap the period analyzed by Smylie et al. (1993a,b) still allowing the possibility that the Slichter triplet was present in the European stack owing to larger excitation. Because of the lack of definite information on these modes either from theory or from observations, one should take into account quite wide ranges of involved frequencies to analyze the probability of the existence of Slichter modes in gravity spectra. Hereafter we analyze, from a statistical point of view, the case of three random peaks appearing in a given frequency window. As already mentioned, the degree of freedom in a specific window $[f_1, f_2]$, where one mode of oscillation is searched, is

$$\frac{f_2 - f_1}{\Delta f}$$

where $\Delta f$ is determined by the time length of the signal. The equivalent random experiment also has to consider the fact that three windows are simultaneously in consideration in the search for the triplet.

The quality factor should also be involved in the discussion. Two major situations may arise:

(1) the length of the signal is rather small, so that the peak width is determined by the length rather than by the damping;

(2) the length of the signal is large so that the width of the peak essentially depends on the $Q$ factor.

In the first part of the paper we did not discuss this point since we assumed case 1. However, when the damping is quite high (small $Q$ factor), the oscillating signal can be shorter than the whole duration of the record, leading to a larger peak width. In this case, the degree of freedom equals the ratio

$$\frac{f_2 - f_1}{\Delta f}$$

where $\Delta f$ is the effective width of the expected peak. Because the $Q$ factor of Slichter modes predicted by theory (anelastic effects) is presently high (e.g. Crossley et al., 1991; Hinderer and Crossley, 1993), we assume hereafter that we are in the undamped case. Nevertheless, for lower $Q$ values like the ones inferred from the claimed triplet (Smylie, 1992), the probabilities we compute hereafter can be easily modified by adjusting the effective degree of freedom $N'$ to the observed width of the individual peaks.

We denote $p$ the probability of exceeding a given level in one window at a given frequency.

Consider now the following example: we have 1 year of hourly samples (8760 points) and we investigate the spectrum in a frequency window of 0.03 cycle $h^{-1}$. The basic frequency interval is $\Delta f = 1/8760 = 0.000114$ cycle $h^{-1}$ and the degree of freedom becomes $N' = 0.03/0.000114 = 262$.

We consider the $2\sigma$, $3\sigma$ and $4\sigma$ threshold values. The corresponding probabilities of exceeding these thresholds are: $p_2 = 0.135355$, $p_3 = 0.011109$ and $p_4 = 0.00033546$.

(1) The probability of having no peak above the level in one window is

$$P = (1 - p)^{N'} \tag{31}$$

and leads to $2.8 \times 10^{-17}$ ($2\sigma$), 0.054 ($3\sigma$) and 0.916 ($4\sigma$).

(2) The probability of having at least one peak above the level in one window is

$$P = 1 - (1 - p)^{N'} \quad (32)$$

and leads to $\cong 1$ ($2\sigma$), 0.946 ($3\sigma$) and 0.084 ($4\sigma$).

(3) The probability of having at least one peak above the level in each of the three windows (same frequency width) is

$$P = \left[1 - (1 - p)^{N'}\right]^3 \quad (33)$$

and gives $\cong 1$ ($2\sigma$), 0.848 ($3\sigma$) and 0.00060 ($4\sigma$).

(4) The probability of having no peak above the level in any of the three windows is

$$P = (1 - p)^{3N'} \quad (34)$$

and gives $2.26 \times 10^{-50}$ ($2\sigma$), 0.000154 ($3\sigma$) and 0.768 ($4\sigma$).

(5) The probability of having at least one peak above the level in one of the three windows is

$$P = 1 - (1 - p)^{3N'} \quad (35)$$

and gives $\cong 1$ ($2\sigma$), 0.99985 ($3\sigma$) and 0.232 ($4\sigma$).

However, it is worth mentioning here that the above discussion only deals with the presence of three random peaks. The three spectral lines corresponding to the translation of the Earth's solid inner core (Slichter modes) are in fact not independent but linked by theoretical relationships forecasting a specific frequency separation (splitting). It is clear that this information can then be used to construct an automated search in the spectrum for correctly split resonance triplets, as shown by Smylie et al. (1993b). We acknowledge that, in this case, the presence of peaks, being no longer random, leads to completely different statistical results to those derived above. Indeed, the product spectrum of gravity in the subtidal band (see Fig. 10 in Smylie et al., 1993b) shows that the three claimed peaks are only slightly emerging from the ambient noise without any statistical evidence (well below the 95% confidence interval), but a test of the splitting parameter reveals that this observational triplet obeying the subseismic law (Smylie, 1992) dominates by far other resonance triplets or other splitting laws with a high degree of significance (see Fig. 12 in Smylie et al., 1993b). Of course, the search is highly dependent on the theoretical frequency separation which is itself still controversial.

### 3.3. Approach with the decision theory

In the previous section, we analyzed the probability for a spectral peak to overpass a given threshold in the presence of random noise. In this probabilistic approach, we have defined a frame in which a spectrum can be investigated and we have concluded that the existence of a wave should not be inferred without due consideration.

To go one step further, one has to answer the following question: assuming the existence of a 'detected' spectral line, what is the probability of having really detected it? This concerns the so-called decision theory (Coulon, 1984). Although this theory should be applied to all the problems raised in this paper, we shall only examine here the simplest case to demonstrate the interest of this approach, postponing deeper analysis to another study.

When a peak is expected at a given frequency, it appears immediately that:

(1) if no peak is visible, the corresponding oscillation has only a weak probability of existence;

(2) if the peak is overcrossing a high amplitude level, for instance $7\sigma$, the probability of existence is very high.

The decision theory, by taking into account the effective observed amplitudes, allows one to evaluate this kind of probability (Arques, 1979).

Define $s_0$ the event: 'no signal is present' and $s_1$ the event: 'the signal is present'. The quantities $s_0$ and $s_1$ are called 'decisions'. In general, the a priori probabilities of these events are unknown.

Two hypotheses may arise: (1) $H_0$, the signal is not present; (2) $H_1$, the signal is present. Consequently, four cases are considered.

(1) One decides $s_1$, $H_0$ being true: this is the false detection. We note $\alpha = P(s_1, H_0)$ the corresponding probability.

(2) One decides $s_0$, $H_0$ being true: this is the true no detection. Its probability is $\alpha' = P(s_0, H_0) = 1 - \alpha$.

(3) One decides $s_0$, $H_1$ being true: this is the false no detection. Let $\beta = P(s_0, H_1)$ its probability.

(4) One decides $s_1$, $H_1$ being true: this is the ultimate true detection. The probability is $\beta' = P(s_1, H_1) = 1 - \beta$.

The measured amplitude $S$ is compared with a threshold $T$. In the absence of the wave, the amplitude follows the Rayleigh law and the probability of false detection is then

$$p_f = \int_T^\infty \frac{r}{\sigma^2} \exp - \left[ \frac{r^2}{2\sigma^2} \right] dr = \exp - \left[ \frac{T^2}{2\sigma^2} \right] \quad (36)$$

Therefore a given probability $\alpha = p_f$ of false detection, the threshold becomes

$$T = \sigma\sqrt{-2 \ln \alpha} \quad (37)$$

Notice that we deal here with two different kinds of threshold: one of probability and one concerning the amplitude value.

In the presence of a wave, the valid statistic is the Rice–Nakagami one and the probability of detection is given by:

$$p_d = \int_T^\infty \frac{r}{\sigma^2} \exp\left( - \frac{r^2 + S^2}{2\sigma^2} \right) I_0(rS/\sigma^2)\, dr = Q(S/\sigma, T/\sigma) \quad (38)$$

The function $Q(a, b)$ is called the Marcum function (Coulon, 1984). It is plotted on Fig. 5 and allows numerical applications. As an example, we assume in the following that the parameter of the Rayleigh law $\sigma$ equals 1. If we fix the threshold $T = 3\sigma$ and observe an amplitude $S = 4$, the probability $p_d$ of having effectively detected a wave is 90%. For $T = 4$ and $S = 3$, the probability is 20%. For $T = 2$ and $S = 2$, we obtain 60%. For $T = 4$ and $S = 4$, the probability drops to 50%.

These two last cases show that if the level has been fixed to a high value, it becomes more difficult to 'detect' a wave. In both cases, the observed amplitude reaches exactly the threshold, but the probabilities are not the same. In other words, if we want to be sure that the wave really exists, we have to determine a high threshold and it becomes naturally more difficult to establish the existence of the wave.

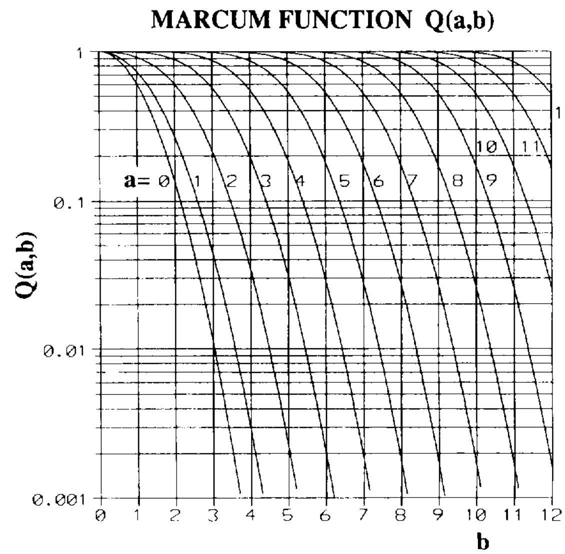


Fig. 5. Marcum function (from Coulon, 1984) allowing computation of the probability of detection according to the decision theory.

## 4. Conclusion

In the first part of this study, we investigated the probability law which occurs, in the presence of random white noise, in the amplitude estimate of a periodic signal using a classical Fourier analysis or any equivalent least-squares approach. A consequence of the asymmetry of the probability law is that there is always a bias between the least-squares and the maximum probability estimates for the amplitude. We have shown that the amplitude is always overestimated and should be corrected, especially for small signal-to-noise ratio and/or when a large number of waves are included in the analysis. There is a direct consequence in tidal analysis for instance, where small waves are used to retrieve the FCN resonance parameters. In the second part, we paid attention to the meaning of the statement 'I think there is a peak appearing at this frequency' and we analyzed this problem with a probabilistic approach. It has been shown that a pure random noise is able to generate, according to the degree of freedom of the spectrum, several peaks that can overpass a specific threshold as large as four times the standard deviation of the noise content. This point was found to be in agreement with the content of a gravity spectrum from a 5 year long record of the French superconducting gravimeter. It is therefore very easy to misinterpret these peaks in terms of true periodic signals. The role played by doubling the number of observations was also investigated in establishing the probability law of the spectral amplitude estimate. We have applied some of the results to the problem of the detection in gravity spectra of the translational triplet of the Earth's inner core (Slichter modes), assuming that the three spectral peaks are independent. Finally, we have introduced the decision theory providing a more mathematical framework for any spectral line research and which should be used in further analyses of this kind.

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