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Bayesian estimation of the free core nutation parameters from the analysis of precise tidal gravity data

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Abstract

This paper is devoted to a new approach to the problem of estimating the free core nutation (FCN) parameters from tidal gravity data. We first review different models of the FCN resonance parameters (period, damping, strength), as well as different inversion techniques used in gravimetry (analytical, linearised least squares, non-linear generalised inverse, stochastic). We propose then a new Bayesian inversion method since such a probabilistic view allows the most complete and reliable information on the FCN resonance to be obtained. We show the consequences of introducing the positivity of the quality factor Q and, even more important, we show that the null information criterion on this factor allows us to better understand why Q is often underestimated in gravity studies (or even found to be negative) with standard least squares techniques. We apply the Bayesian estimation method to a set of gravity data originating from a 3000-day record (1988–1996) of the superconducting gravimeter (SG T005) near Strasbourg (France), after correction for pressure effects and ocean loading with the help of a recent model derived from satellite altimetry. The marginal distribution found for the eigenperiod is nearly Gaussian and leads to a most probable value of 428 days, in agreement with previous gravity studies. The Q distribution is found to be highly asymmetrical with a flat maximum probability that Q exceeds 10^5, in agreement with the high values derived from very large baseline interferometry (VLBI) studies. The joint probability laws of the parameters show that strong correlations exist between some couples of parameters. Finally, the important impact of inaccurate ocean loading corrections on the determination of the damping of the eigenmode is pointed out; it appears that wrong ocean corrections may lead to underestimation of Q or, even worse, to generation of negative Q when using standard least squares retrieval techniques applied to gravity data.

Keywords: Bayesian; Free core nutation; Tidal gravity; Superconducting gravimeter; Ocean loading

1. Introduction

Because of the fluidity of the core, the Earth has, in addition to the Chandler wobble of 435-day period (in a rotating geographical frame), another rotational mode called free core nutation (FCN) of nearly diurnal period (in the same frame). This mode is physically due to the pressure coupling between the liquid core and the solid mantle which acts as a restoring force. Its eigenperiod directly depends on the core–mantle boundary (CMB) ellipticity, as well as on the Earth's elasticity (see Toomre, 1974; Sasao et al., 1980; Sasao and Wahr, 1981). Its observation

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is therefore a very important means to study the Earth's deep interior dynamics and structure. The attenuation of the mode (defined by a quality factor) is also important because it is a direct consequence of damping mechanisms inside the Earth. There are basically two techniques to observe the FCN: on one hand, very large baseline interferometry (VLBI), which is able to measure the amplitude of the mode (and its damping) in an inertial frame as well as the amplitude and phase of the lunisolar nutations which are altered by the presence of this mode by a resonance process; on the other hand, precise tidal gravimetry (especially with the use of superconducting gravimeters) where some diurnal tides of frequency close to the eigenfrequency are exhibiting amplitude and phase perturbations, again due to the resonance process. Numerous studies have used one of these techniques (or a stack of both) to retrieve the FCN parameters. It is usually accepted that there is a fairly good convergence for the eigenperiod which differs from the theoretically predicted value by about 30 sidereal days. However, the quality factor values from gravimetry are basically often smaller (sometimes even negative Q are found) typically by a factor 10 than the Q inferred from VLBI. A summary of the estimates from different datasets and techniques (spring and superconducting gravimetry, VLBI) is given in Table 1, where we have also added theoretical results relative to an elastic Earth (Sasao et al., 1980) and to a slightly inelastic one (Wahr and Bergen, 1986). In this paper, we will focus on the gravimetry results and try to understand the reasons for this discrepancy in damping estimate. We will first review the different models which are usually considered to express the resonance (Section 2). Section 3 will be devoted to a review of the different methods to solve the problem according to a given model and we will introduce the Bayesian formulation. The results from this new approach are provided using, e.g., a 3000-day long record from the superconducting gravimeter GWR T005 located near Strasbourg. In Section 4, we will point out the effect of inaccurate ocean loading corrections in the FCN inversion problem.

2. A review of different models for the FCN resonance

2.1. Use of a reference gravimetric factor in the resonance model

The basic equation which holds for describing the resonance model in tidal gravity can be written in the following form (Hinderer et al., 1991a):

$$\tilde{\delta}_j = \tilde{\delta}_0 + \frac{\tilde{a}}{\sigma_j - \tilde{\sigma}_{\mathrm{nd}}}, \tag{1}$$

where $\tilde{\delta}_j$ is the complex gravimetric factor (observed data) for every tidal wave of frequency σ_j; this factor

Table 1
A summary of various estimates of period and quality factor of the FCN
In addition to theoretical results relative to an elastic Earth and to a slightly inelastic one, we have added experimental results from the International Digital Accelerometers (IDA) network of spring gravimeters and from VLBI. The other results are from superconducting gravimeter (SG) datasets: B = Brussels (Belgium), BH = Bad Homburg (Germany), ST = Strasbourg (France), CA = Cantley (Canada), J = three Japanese stations.

Author	Data	T	Q
Neuberg et al. (1987)	stacked gravity B + BH	431 ± 6	2800 ± 500
Sasao et al. (1980)	theory elastic	465	∞
Wahr and Bergen (1986)	theory inelastic	474	78 000
Herring et al. (1986)	VLBI	435 ± 1	22 000–100 000
Cummins and Wahr (1993)	stacked gravity IDA	428 ± 12	3300–37 000
Sato et al. (1994)	stacked gravity J	437 ± 15	3200–∞
Defraigne et al. (1994)	stacked gravity B + BH + ST	425 ± 6	2400–8300
Florsch et al. (1994)	gravity J9	431 ± 1	1700–2500
Merriam (1994)	gravity CA	430 ± 4	5500–10 000
Hinderer et al. (1995)	stacked gravity J9 + CA	429 ± 8	7700–∞

allows us to express the gravity changes at the Earth's surface $(r=a)$ $\tilde{\Delta} g_j(\sigma_j,a) = -\tilde{\delta}_j \frac{2W_j(\sigma_j)}{a}$, as a function of the tidal potential W_j; this near-resonant approximation has been used by Zürn et al. (1986) and Florsch et al. (1994) for instance.

The unknowns to be determined are then the reference gravimetric factor $\tilde{\delta}_0 = \delta_0^{\rm R} + i\delta_0^{\rm J}$, the strength $\tilde{a} = a^{\rm R} + ia^{\rm I}$ and $\tilde{\sigma}_{\rm nd} = \sigma_{\rm nd}^{\rm R} + i\sigma_{\rm nd}^{\rm I}$, which is the complex eigenfrequency of the FCN. $\tilde{\delta}_0$ would be the value of the gravimetric factor without any resonance process; it is also the asymptotic value of $\tilde{\delta}_j$ for frequencies far away from the resonance eigenfrequency. In a first approximation, one can state that it is the measured value for a wave of known amplitude and quite distant frequency like O_1. However, to infer the error introduced by this approximation, one has to estimate the covariance between this term and the other unknowns; if there is none between $\tilde{\delta}_0$ and $\tilde{\sigma}_{\rm nd}$ for instance, then and only then the choice of a given value for $\tilde{\delta}_0$ will not interfere with the determination of $\tilde{\sigma}_{\rm nd}$. It is therefore needed to perform the inversion by taking $\tilde{\delta}_0$ into account as an unknown in order to estimate the potential covariances; an example is shown later where a correlation indeed exists between $\delta_0^{\rm R}$ and $\sigma_{\rm nd}^{\rm R}$.

Two quantities are of special interest in geodynamics: on one hand, the eigenperiod T (expressed hereafter in sidereal days in the rotating frame) of the FCN related to $\sigma_{\rm nd}^{\rm R}$ by:

$$T = \frac{2\pi}{\sigma_{\rm nd}^{\rm R}}, \tag{2}$$

where $\sigma_{\rm nd}^{\rm R}$ is expressed in rad/sidereal day.

Another notation sometimes found for the eigenperiod T' (expressed in sidereal days in inertial space) is:

$$T' = \frac{1}{\sigma_{\rm nd}^{\rm R} C - 1}, \tag{3}$$

if $\sigma_{\rm nd}^{\rm R}$ is expressed in deg/solar hour ($C = 86\,164/15 \times 86\,400$); on the other hand, the quality factor Q expressing the damping due to all physical processes involved in the resonance is defined as $Q = \sigma_{\rm nd}^{\rm R}/2\sigma_{\rm nd}^{\rm I}$.

2.2. Damped oscillator approximation

The following physical model has been used in several previous studies (e.g., Neuberg et al., 1987; Hinderer et al., 1991b):

$$\tilde{\delta}_j = \tilde{\delta}_0 + \frac{-2\Omega\tilde{a}}{\sigma_j^2 - \sigma_0^2 - 2i\sigma_j\sigma_{\rm nd}^{\rm I}}, \tag{4}$$

with

$$\sigma_0^2 = \left(\sigma_{\rm nd}^{\rm R}\right)^2 + \left(\sigma_{\rm nd}^{\rm I}\right)^2$$

This model is only an approximation of the true resonance Eq. (1), but was introduced by analogy with the classical damped oscillator theory in physics. It is, however, interesting to compare both models (Eqs. (1) and (4)) in this review in order to validate the previous FCN studies based on this approach:

$$\begin{cases} \sigma_j = -\Omega(1+\varepsilon_j) \\ \sigma_{\rm nd}^{\rm R} = -\Omega(1+\varepsilon^{\rm R}), \\ \sigma_{\rm nd}^{\rm I} = -\Omega\varepsilon^{\rm I} \end{cases}$$

and assuming that ε_j, $\varepsilon^{\rm R}$, $\varepsilon^{\rm I}$ are small first-order quantities, one easily obtains:

$$\begin{aligned} \tilde{\delta}_j &= \tilde{\delta}_0 + \frac{\dfrac{-2\tilde{a}}{\Omega}}{(1+\varepsilon_j)^2 - (1+2\varepsilon^{\rm R}) - 2i(1+\varepsilon_j)\varepsilon^{\rm I}} \\ &= \tilde{\delta}_0 - \frac{\tilde{a}}{\Omega\left[(\varepsilon_j+1) - (\varepsilon^{\rm R}+1) - i\varepsilon^{\rm I}\right]} \\ &= \tilde{\delta}_0 + \frac{\tilde{a}}{\sigma_j - \tilde{\sigma}_{\rm nd}}. \end{aligned} \tag{5}$$

The condition $\varepsilon_j \ll 1$ requires tidal waves with frequencies close to one sidereal day which is indeed the case for waves P_1, K_1, Ψ_1 and ϕ_1 but is less valid for O_1 where $\varepsilon_j = 1/13.7$. We also have $\varepsilon^{\rm R} \cong (1/435) \ll 1$, as indicated by previous inversion results (see, e.g., Defraigne et al., 1994) for the discrepancy between the FCN eigenperiod and the sidereal day, as well as $\varepsilon' \ll 1$, if $(1+\varepsilon^{\rm R}/2Q) \approx (1/2Q) \ll 1$ which is also true for the large values of the quality factor (several thousands) retrieved from the same previous investigation. In conclusion, no large misfit was caused by this approximation and the results found in the literature based on the damped harmonic oscillator are therefore comparable to the ones derived from the true resonance model we will use hereafter.

2.3. Positivity of the quality factor and final formulation

The quality factor previously defined depends on σ_{nd}^{R} and σ_{nd}^{I}, both terms being usually retrieved from the resonance. It should be noted that the real and imaginary parts of the gravimetric factor follow Gaussian distributions, in agreement with the central limit theorem applied to Fourier coefficients, while the amplitude of the complex gravimetric factor follows a Rice–Nakagawi distribution (Florsch et al., 1995). The two parameters, σ_{nd}^{R} and σ_{nd}^{I}, also seem to follow a Gaussian distribution, as demonstrated by a stochastic approach (Hinderer et al., 1994b) reviewed in Section 3.5. Actually, the dependence of σ_{nd}^{I} with the imaginary part of the gravimetric factors (see Eq. (13) below) can lead to possible negative values of σ_{nd}^{I} because of inaccurate ocean loading corrections while this parameter should be positive by nature (the phase lag observed in the tides having a physical meaning).

The important question which arises is: What happens to the distribution law for Q itself? Indeed, when a parameter is normally distributed, the solution is simply defined by two quantities (mean value and standard deviation) and different inversion methods easily converge; however, if it is not the case, like here for Q which consists in a ratio (see below), then the problem is more complicated. If assuming first that σ_{nd}^{R} and σ_{nd}^{I} follow normal probability distributions (true if no positivity constraint is applied), the corresponding law for Q is given by a very complicated expression which leads to unlikely Q values. A synthetic example deriving from realistic values for σ_{nd}^{R} and σ_{nd}^{I} and their errors is shown on Fig. 1; one can observe that Q has a very

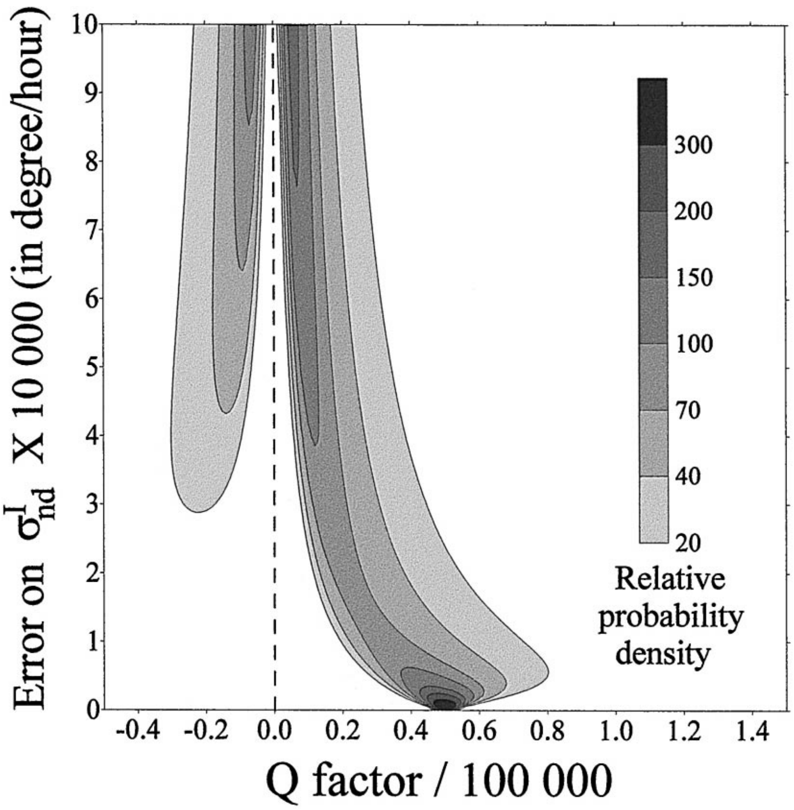

Fig. 1. PDF of the quality factor Q for increasing errors $\Delta\sigma_{nd}^{I}$; notice on the left side the appearance of negative Q values for large errors.

asymmetrical bi-modal distribution with possible negative values, as indicated on the left part of the plot, for sufficiently large value of the error $\Delta\sigma_{nd}^{I}$.

Another example using a Monte Carlo inversion for Q starting from real gravity data gave similar results (see Fig. 2 in Hinderer, 1997). The parameter σ_{nd}^{R} is present in the numerator of Q and has usually an associated error much smaller than the value itself; hence, negative values are impossible (nearly zero probability of appearance). In a first approximation, the Q distribution is therefore the product of the σ_{nd}^{R} distribution with the $1/\sigma_{nd}^{I}$ distribution. In contrast, the error on σ_{nd}^{I} can easily be of the same order as the value itself and leads then to negative values; as a consequence, the quality factor Q is possibly also negative or even tends to infinity (for values of σ_{nd}^{I} close to 0). The zone where the Q law reaches a maximum for negative Q is rather distant from the origin because small negative Q values would imply large negative σ_{nd}^{I} values which have a low probability of occurrence. This explains the bi-modal shape of the Q distribution law. All these results indicate that these approaches are not correct, and that the positivity of σ_{nd}^{R} and σ_{nd}^{I} *by nature* has to be introduced in the formalism.

Indeed, all quantities involved in the Q ratio are positive by definition; therefore, they should rather follow a log-normal distribution law (see Tarantola, 1987) to avoid possible negative values in a Gaussian normal law, which then would lead to a non-physical negative Q. The random additional noises appearing during the measurement procedure lead indeed to Gaussian probability density function (PDF) which are not realistic. When estimating such a parameter, one should include the a priori information that Q is positive.

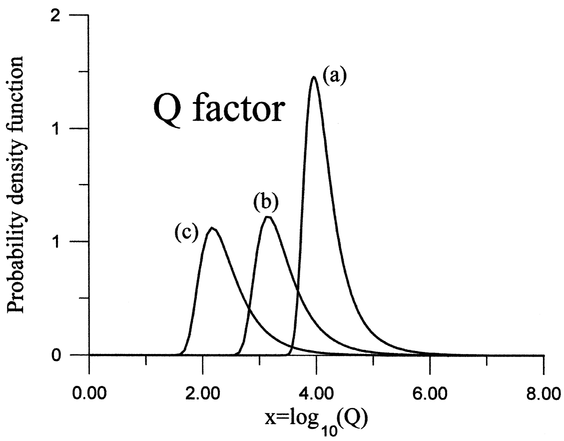

Fig. 2. PDF for $x = \log(Q)$, when positivity is imposed ($\sigma_{nd}^{I} > 0$), assuming the same mean value for σ_{nd}^{I} and increasing error estimate $\Delta\sigma_{nd}^{I}$ (the value increases by a factor 10 from case a to case b, and from case b to case c). The decrease of Q with increasing errors is unreasonable and excludes high values.

It is therefore better to take into account a priori the positivity of σ_{nd}^{I} in the model or, equivalently, of Q, which is the sought parameter for damping in FCN retrieval studies.

Introducing a change in the variable $Q \to x$:

$$Q = \frac{\sigma_{nd}^{R}}{2\sigma_{nd}^{I}} = 10^{x},$$

and inverting $x = \log_{10}(Q)$ instead of Q insures the positivity of Q (and of σ_{nd}^{I}).

A further advantage of this variable change is to introduce a kind of linearisation in the problem. It is the same natural scale to change Q from 100 to 1000 than from 1000 to 10 000. In fact, as discussed below in the Bayesian inversion, it will enable us to introduce more simply the concept of null information on Q (lack of information on the physical damping amplitude).

The equations relative to the resonance model become then:

$$\begin{cases} \delta_j^{R} = \delta_0^{R} + \dfrac{a^{R}\left(\sigma_j - \sigma_{nd}^{R}\right) - \dfrac{a^{I}\sigma_{nd}^{R}10^{-x}}{2}}{\left(\sigma_j - \sigma_{nd}^{R}\right)^2 + \left(\dfrac{\sigma_{nd}^{R}10^{-x}}{2}\right)^2} \\ \delta_j^{I} = \delta_0^{I} + \dfrac{a^{R}\left(\sigma_j - \sigma_{nd}^{R}\right) + \dfrac{a^{R}\sigma_{nd}^{R}10^{-x}}{2}}{\left(\sigma_j - \sigma_{nd}^{R}\right)^2 + \left(\dfrac{\sigma_{nd}^{R}10^{-x}}{2}\right)^2} \end{cases} . \qquad (6)$$

This will be our final formulation for the resonance model. Notice that it is not required, in practice, to introduce the positivity of σ_{nd}^{R} since the error of this parameter is much smaller that its own value, leading to the coincidence of the Gaussian law with the log-normal one in that case.

There is still a problem left: as shown by Fig. 2, assuming the same mean value for σ_{nd}^{I} and increasing the errors lead to PDF where the maximum probability value for x, or equivalently for Q, decreases; this unreasonable behaviour hence excludes high Q values. The problem will be solved in the Bayesian approach once the null information concept on the damping has been properly introduced (see Section 3.6).

3. A review of different methodologies applied in the resonance inversion

3.1. Generalities

In their synthesis article, Tarantola and Valette (1982a) state that the available information from physical measurements is fundamentally probabilistic and that inverting data means propagating this probabilistic information from the data (most of the time assuming Gaussian distribution laws) to the parameters via a physical relationship which expresses the data as a function of the parameters (model). The result from the inversion is hence the probability law of the parameters itself, from which one can compute by integration the probability that a given parameter ranges within a given interval. With respect to this approach, notice that the classical mean squares method provides only the mean value of the parameters and that the generalised inverse method (Tarantola and Valette, 1982b) provides only the most probable value. In some cases, these two quantities do not represent the true solution, especially in the case of strongly asymmetric or multimodal laws.

Before introducing the Bayesian approach, we first review hereafter different approaches which have been used in the past to retrieve the FCN parameters.

3.2. Analytical solution

Using the model described by Eq. (1), one has to determine three complex quantities: $\tilde{\delta}_0$, $\tilde{a}$, and $\tilde{\sigma}_{nd}$ (i.e., six real values). Therefore, any combination of three tidal waves (six parameters: three amplitudes and three phases) is, in principle, sufficient to analytically find the solution, as demonstrated by Florsch et al. (1994):

$$\begin{cases} \tilde{\delta}_0 = \dfrac{\sigma_1\tilde{\delta}_1(\tilde{\delta}_3-\tilde{\delta}_2)+\sigma_2\tilde{\delta}_2(\tilde{\delta}_1-\tilde{\delta}_3)+\sigma_3\tilde{\delta}_3(\tilde{\delta}_2-\tilde{\delta}_1)}{\sigma_1(\tilde{\delta}_3-\tilde{\delta}_2)+\sigma_2(\tilde{\delta}_1-\tilde{\delta}_3)+\sigma_3(\tilde{\delta}_2-\tilde{\delta}_1)} \\ \tilde{\sigma}_{nd} = \dfrac{\sigma_1\tilde{\delta}_1(\sigma_2-\sigma_3)+\sigma_2\tilde{\delta}_2(\sigma_3-\sigma_1)+\sigma_3\tilde{\delta}_3(\sigma_1-\sigma_2)}{\sigma_1(\tilde{\delta}_3-\tilde{\delta}_2)+\sigma_2(\tilde{\delta}_1-\tilde{\delta}_3)+\sigma_3(\tilde{\delta}_2-\tilde{\delta}_1)} \\ \tilde{a} = -\dfrac{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)(\sigma_3-\sigma_1)(\tilde{\delta}_1-\tilde{\delta}_2)(\tilde{\delta}_2-\tilde{\delta}_3)(\tilde{\delta}_3-\tilde{\delta}_1)}{\left[\sigma_1(\tilde{\delta}_3-\tilde{\delta}_2)+\sigma_2(\tilde{\delta}_1-\tilde{\delta}_3)+\sigma_3(\tilde{\delta}_2-\tilde{\delta}_1)\right]^2} \end{cases}, \tag{7}$$

where typographical errors have been corrected ($\tilde{\delta}_i$ is the complex gravimetric factor for every of the three waves $i = 1, 2, 3$). The authors suggested the use of waves O_1, K_1 and Ψ_1, for their proximity to the resonance or their precise determination after tidal analysis and ocean loading correction. Obviously, a number of three wave triplets can be selected and the results may vary according to the choice; a further disadvantage was the lack of realistic error estimates. It is better to use a global adjustment to incorporate more information. One interest of the analytical formulation is however to show precisely the parameter dependence with the data.

3.3. Linearised least squares estimation

When more than three waves are considered, a least squares method can be applied. The problem then becomes an optimisation problem in determining the resonance parameters by minimising the functional:

$$J\left(\tilde{a},\tilde{\delta}_0,\tilde{\sigma}_{nd}\right) = \sum_j \left|\tilde{\delta}_j - \left(\tilde{\delta}_0 + \frac{\tilde{a}}{\sigma_j - \tilde{\sigma}_{nd}}\right)\right|^2. \tag{8}$$

This method has been widely used in previous FCN retrieval studies like in Neuberg et al. (1987), Richter and Zürn (1986), Zürn and Rydelek (1991), and Defraigne et al. (1994), Defraigne et al. (1995). These authors adopted, most of the time, the damped harmonic oscillator model (without reference gravimetric factor) and used the Marquardt (1963) optimisation algorithm. This method searches for the mean of the parameters and requires a starting model to initiate the iteration scheme.

3.4. Generalised non-linear inverse solution

In this method, one searches for the maximum probability of occurrence of the parameters taking a priori information into account. The recursive formula for this method is given by Tarantola and Valette (1982b):

$$\hat{p}_{k+1} = p_0 + \mathbf{C}_{p_0 p_0}\mathbf{G}_k^{\mathrm{T}}\left(\mathbf{C}_{\boldsymbol{d}_0\boldsymbol{d}_0} + \mathbf{G}_k\mathbf{C}_{p_0 p_0}\mathbf{G}_k^{\mathrm{T}}\right)^{-1} \times\left[\boldsymbol{d}_0 - g(\hat{p}_k) + \mathbf{G}_k(\hat{p}_k - p_0)\right], \quad (9)$$

where p_0 is the initial estimation for the parameter vector $\boldsymbol{p}$; $\hat{p}_k$ the estimated value at iteration step k; $\mathbf{C}_{p_0p_0}$ the a priori covariance matrix of the parameters (often diagonal by lack of knowledge); $\mathbf{C}_{\boldsymbol{d}_0\boldsymbol{d}_0}$ the covariance matrix of the data (also diagonal if the data are independent); $\mathbf{G}_k$ and $\mathbf{G}_k^{\mathrm{T}}$ the derivative matrix of the data with respect to the parameters and its transpositions; and $\boldsymbol{d}_0$ the data vector.

The two following hypotheses are made:

- data are normally distributed (Gaussian PDF); and
- the model is exact.

The first hypothesis is clearly valid for gravity data (Florsch et al., 1991, 1995; Hinderer et al., 1993, 1994a, 1995; Jensen et al., 1995), but the second one is much less obvious because the resonance model takes only imperfectly into account ocean and atmospheric loading effects as correcting terms (e.g., Crossley et al., 1995; Melchior and Francis, 1996). However, the errors in these corrections can be assumed to be normally distributed and the second hypothesis holds if one introduces these errors in the corrections in addition to the formal errors originating from the tidal analysis (in fact, this increases the uncertainties quite considerably).

One difficulty of the method is the existence of local minima. To avoid this, the a priori variances on the parameters have to be lowered or positivity constraints have to be introduced on some parameters via an adequate variable change.

3.5. Stochastic inversion

The previous inversion method associates data values to parameter values. It is hence possible to scan the parameter space and to obtain the probability laws. We note $\boldsymbol{d} = g(\boldsymbol{m})$ the direct problem and $\boldsymbol{m} = h(\boldsymbol{d})$ the inverse problem. It is possible to randomly generate Gaussian data $\boldsymbol{d}$ centred on a mean value $\boldsymbol{d}$ with a standard deviation $\Delta\boldsymbol{d}$. For every dataset, the inversion leads to a parameter set $\boldsymbol{m}$ and the distribution histogram can be computed. These Monte Carlo-type simulations have been used previously (Hinderer et al., 1994b; Hinderer, 1997). In this last study, a Monte Carlo simulation was performed by generating a large number (10^6) of datasets taken randomly distributed around observed data from a 2-year long registration from the French and Canadian superconducting gravimeters. The histograms showing the a posteriori PDF of the parameters indicated that substantial information could be gained from the observations. Another interesting outcome was the computation of the joint probability distributions between some couple of parameters indicating either weak correlation (e.g., between real strength and imaginary eigenfrequency) or strong correlation (e.g., between real strength and real eigenfrequency) (see also Cummins and Wahr, 1993; Defraigne et al., 1994).

This stochastic method based on Monte Carlo simulations is still not the best one. Indeed, the propagation of the information, which is present in the data to the parameter space, requires an inversion procedure which does not fully exploit the available information (each inversion leads to one set of parameters).

3.6. Bayesian inversion

The Bayesian inversion is the method which best propagates the data information to the parameters. The result is indeed the knowledge of the probability law for each parameter, as shown by Tarantola and Valette (1982a) (Eq. 6-4):

$$\sigma_{\mathrm{p}}(\boldsymbol{p}) = \int \frac{\rho(\boldsymbol{d},\boldsymbol{p})\,\theta(\boldsymbol{d},\boldsymbol{p})}{\mu(\boldsymbol{d},\boldsymbol{p})}\mathrm{d}\boldsymbol{d}, \quad (10)$$

where: $\sigma_{\mathrm{p}}(\boldsymbol{p})$ is the a posteriori probability density for the parameter $\boldsymbol{p}$; $\rho(\boldsymbol{d},\boldsymbol{p})$ the a priori joint probability density for the data $\boldsymbol{d}$ and parameter $\boldsymbol{p}$, $\theta(\boldsymbol{d},\boldsymbol{p})$ the probability density of the model; and $\mu(\boldsymbol{d},\boldsymbol{p})$ the null information criterion (full ignorance) on both data and parameters.

We will consider the following additional hypotheses.

(a) There is an explicit relationship $\boldsymbol{d} = g(\boldsymbol{m})$, where $\boldsymbol{d}$ is the data vector and $\boldsymbol{m}$ the parameter vector to be sought; in our case, this is provided by Eq. (8).

(b) The null information $\mu(\boldsymbol{d},\boldsymbol{p})$ is taken constant; this question is not trivial and Tarantola and Valette (1982a) propose to define the null information on a parameter from the parameterisation invariance with respect to a group transformation. In our case, the positivity of the quality factor and its logarithmic character lead to state that the null information criterion for Q is $\mu(x) = \mu(\log_{10} Q) = \text{const.}$ (it is equivalent to set $\mu(Q) = \text{const.}/Q$).

(c) The relationship $\boldsymbol{d} = g(\boldsymbol{m})$ is assumed to be exact; as already said, it is only approximately true because of the imperfectly modelled contribution originating from the ocean loading. Two options are possible: either state that the relationship is not exact and take this point into account in the inversion (assuming then precise ocean loading corrections) or assume an exact model but introduce measurements errors which are compatible with the precision of the ocean corrections. The second choice is more convenient because the computation of the a posteriori law can be achieved more easily.

(d) Data are assumed to be independent from the parameters; it is simply stating that the measurement itself is without action on the collected data.

(e) The a priori law on the parameters is taken uniformly on a specific interval; this is equivalent to accepting full ignorance within a given range of values.

(f) The data follow a Gaussian normal law and are independent (diagonal covariance matrix).

In this case, the probability density laws for the parameters can be computed using:

$$P(\boldsymbol{p}) = P\left(\tilde{\delta}_0, \tilde{a}, \tilde{\sigma}_{\text{nd}}\right) = k \exp\left\{ -\frac{1}{2} \sum_j \left[\left(\frac{\delta_j^{\text{Rt}} - \delta_j^{\text{Rm}}}{\Delta \delta_j^{\text{Rm}}} \right)^2 + \left(\frac{\delta_j^{\text{It}} - \delta_j^{\text{Im}}}{\Delta \delta_j^{\text{Im}}} \right)^2 \right] \right\}, \quad (11)$$

where k is a normalisation factor in order than the integral of Eq. (11) is unity; R is for the real part and I for the imaginary part, m holds for the measurement value and t for the theoretical value (depending on the sought parameters), $\Delta\delta$ is the error on δ (standard deviation).

From the computational point of view, one has to compute expression (11) by scanning (on a regular grid for instance) all possible parameters. A set of six parameters (complex reference factor, complex strength, complex eigenfrequency) defined on a 100-point grid leads to $100^6 = 10^{12}$ computations! It is therefore clear that such an approach can only be used for a limited number of parameters and grid points. An alternative solution is to use a Markov process (see Roussignol et al., 1993), but then the resulting probability laws are somehow less 'cosmetic'.

The previous formula gives the general probability laws for the parameter vector. In order to obtain the law for one or two parameters, marginal probability laws have to be computed.

The program structure avoids a systematic storage of all (N^m), where N is the number of exploration points in the interval $[p_{\min}, p_{\max}]$ and m is the number of parameters. However, the computation has to be redone for every couple of parameters.

We first applied a classical inversion technique in order to have a first guess of the parameters (p) of the solution and the errors Δp. The starting data are the complex gravimetric factors inferred from a tidal analysis of a 3000-day record (1988–1996) originating from the superconducting gravimeter GWR T005 located near Strasbourg (see Hinderer et al., 1998); these data are corrected for ocean loading according to model CSR3.0 (with mass conservation taken into account), which is one of the most performing global models derived from satellite altimetry (Andersen et al., 1995) (see also Melchior and Francis, 1996; Llubes and Mazzega, 1997); these data have also been corrected for atmospheric pressure effects with the help of a classical barometric admittance (Crossley et al., 1995).

In a second step, the exploration is done on the six-parameter set (three real and three imaginary parts of $\tilde{\delta}_0$, $\tilde{a}$ and $\tilde{\sigma}_{\text{nd}}$). The explored space is hence defined as the Cartesian product of the intervals containing the six parameters:

$$\Pi_j\left[p_j - k\Delta p_j, p_j + k\Delta p_j \right].$$

This first exploration shows that the imaginary part of the strength a^{I} is close to 0 with an error much larger than the value itself (see Fig. 3). We will hence cancel this parameter which restricts the scanning to a five-parameter space with 50 values per parameter leading to 50^5 values. As already said, the results using the Bayesian approach are the probability laws themselves. As these cannot be stored simultaneously on a physical system, we have performed the computation of the joint probability density laws directly in the code for all couples of parameters. They are plotted on Fig. 4. One can easily distinguish strong correlated parameters (tilted elliptical shape) from almost uncorrelated ones (untilted shape). As expected from the expression of the model, the real parts of the parameters are, in general, highly correlated one to the other (and similarly for the imaginary parts) while real and imaginary parts are weakly correlated.

A further integration of the PDF leads to the marginal probability for each of the parameters and the special case of the eigenperiod T and quality factor Q (or more precisely $x = \log_{10} Q$) is given by Fig. 5. In the general case of highly non-linear problems with a large number of model parameters, the inspection of marginal probabilities like done here is impractical; one has then to use a more efficient Monte Carlo sampling method of the solutions in order to convey useful information on the model properties as shown by Mosegaard and Tarantola (1995).

As expected, the histogram for T is quite Gaussian and has a most probable value close to 428 sidereal days. This value is in agreement with results found previously in tidal gravimetry (see Table 1). The Q distribution is highly asymmetrical with a flat maximum probability value from $x = 5$ to infinity. The probability that x is between the interval [6,8] (i.e., 10^6 to 10^8 for Q) rather than [5.0,5.2] (i.e., 100 000 to 158 000 for Q) is much larger. Small values of $Q < 10\,000$, as sometimes quoted in gravity studies (see Hinderer, 1997), are therefore highly unprobable; on the contrary, high Q values as seen in VLBI of the order of several tens of thousands (see, e.g., Herring et al., 1986; Defraigne et al., 1994, 1995) are very easily reachable in our inversion method of tidal gravimetric data, provided that the null information concept has been properly introduced in the inversion procedure.

It also points out the non-sense of attributing a single value to Q as well as a given uncertainty; the solution is the full probability density law. Notice that it would have been possible to perform the computation by introducing $\sigma^{\mathrm{I}}_{\mathrm{nd}}$ instead of $x =$

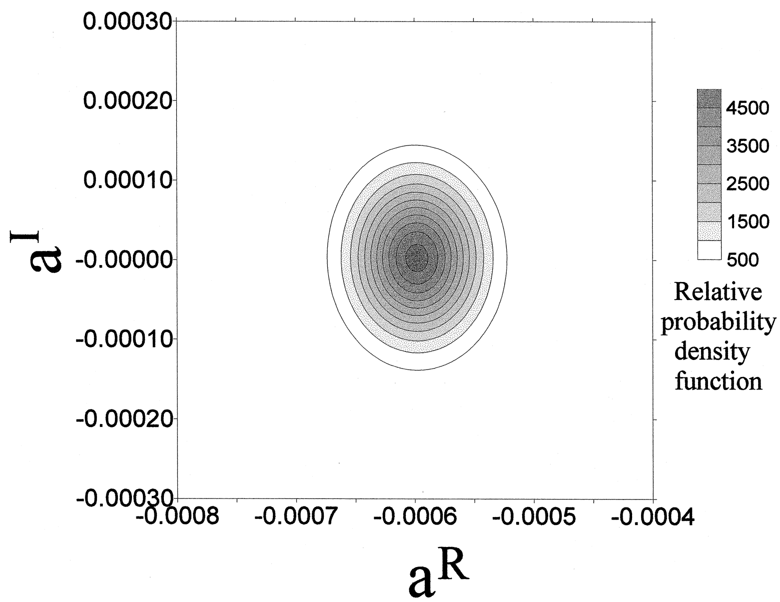

Fig. 3. Systematic exploration of the parameter space pointing out that the imaginary part of the strength is nearly zero; this term will therefore be cancelled in the Bayesian approach.

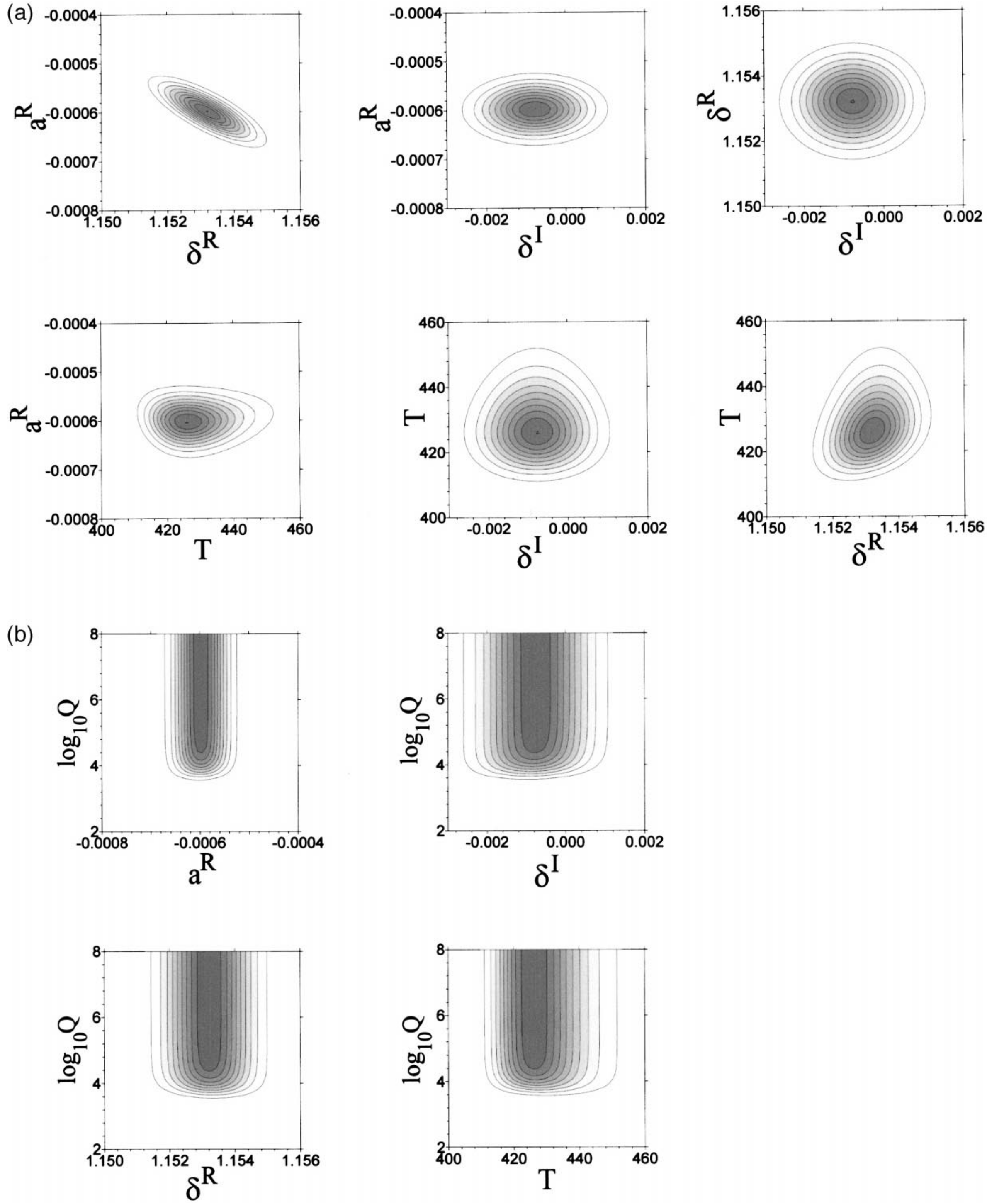

Fig. 4. (a) Joint PDFs for the FCN parameters (real part of strength, real and imaginary parts of reference gravimetric factor, eigenperiod T); the nearly circular (untilted) shapes indicate weak correlations on the contrary to the elliptical (tilted) shapes. (b) Joint PDFs involving the FCN Q factor; notice that large values (even infinite) are possible (tailed shape of the distribution).

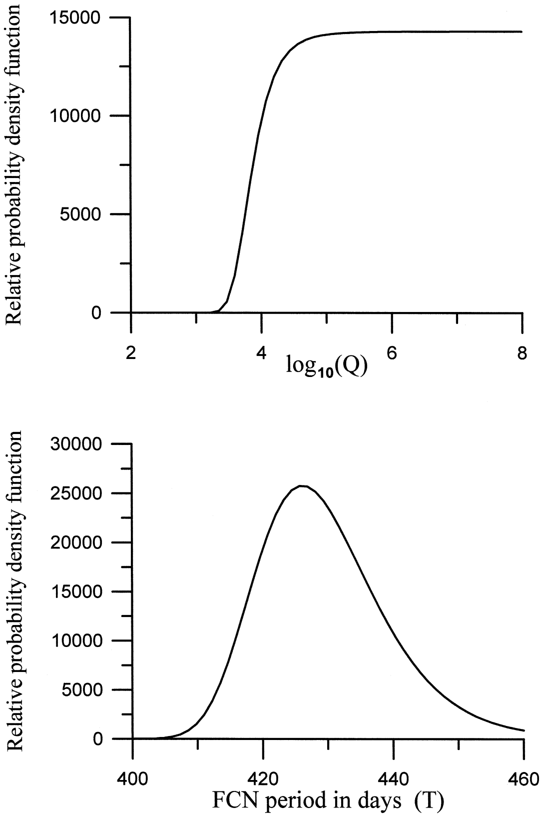

Fig. 5. Marginal probability laws for the eigenperiod T and quality factor Q. Note the highly asymmetric shape of the Q distribution indicating a high probability of Q values larger than 10^5.

$\log_{10} Q$. In that case, the proper 'null information' density function would have been $\mu(\sigma_{nd}^{I}) = (1/\sigma_{nd}^{I})$ and not $\mu(\sigma_{nd}^{I}) =$ const., as implicitly considered in classical inversion methods.

Another interesting consequence of the Bayesian approach when the null information criterion on the damping is taken into account ($\mu(x) = \mu(\log_{10} Q)$ = const.), is shown on Fig. 6. Contrary to Figs. 1 and 2, it is noticeable that the most probable value remains unshifted whatever the error estimate on σ_{nd}^{I} may be.

4. Estimate of the quality factor and its link to ocean loading effects

Most of the quality factor estimations from gravimetry data (especially from superconducting gravimeter data) led to values that the authors themselves considered underevaluated. Sometimes, even negative Q were obtained from standard linearised least squares methods, as shown by Hinderer (1997). Other estimates from space geodesy (VLBI) provided values higher than those derived from the gravity data (see, e.g., Herring et al., 1986).

We have shown that a proper inversion method introducing the null information on the damping (and implicitly the positivity of Q) allows us to find a Q a posteriori distribution with preferred values larger than 10^5 which are easily compatible with the values derived from VLBI. We believe that an additional reason for the remaining discrepancies in Q estimates lies in the lack of accurate ocean loading corrections of the gravity data (the VLBI data also need to be corrected for ocean loading but the correction is not station-dependent as for ground-based gravity measurements).

σ_{nd}^{I} is generally small, only slightly larger than its own error (according to expected high Q values). From resonance expression (1), one obtains, after some elementary calculations and taking $\tilde{a} = a$ as real (this is however not restrictive):

$$\sigma_{nd}^{I} = \frac{a\left(\delta_j^{I} - \delta_0^{I}\right)}{\left(\delta_j^{R} - \delta_0^{R}\right)^2 + \left(\delta_j^{I} - \delta_0^{I}\right)^2}. \qquad (12)$$

In any case, one has $(\delta_j^{I} - \delta_0^{I})^2 \ll (\delta_j^{R} - \delta_0^{R})^2$. Noting $K = a/(\delta_j^{R} - \delta_0^{R})^2$, and $\varepsilon = \delta_j^{I} - \delta_0^{I}$, leads to:

$$\sigma_{nd}^{I} \cong K\left(\delta_j^{I} - \delta_0^{I}\right) = K\varepsilon. \qquad (13)$$

This relation exhibits the nature of the dependence of σ_{nd}^{I} (and hence of Q) with respect to the imaginary part of the (complex) delta factors (or, in other terms, to the phase delay of the tidal waves). Theoretically the parameter ε is expected to be very small and positive. However, the measurement of ε is affected by the oceanic loading. We denote by C this loading contribution (exact, but unknown). The correction uses an estimate, say C', so that, after correction, the following value is used in the inversion: $\varepsilon' = \varepsilon + C - C'$. But, if $|C - C'| \gg |\varepsilon|$ (i.e., the error in the ocean loading correction is large compared to the measurement ε), we have $|\varepsilon'| = |\varepsilon + C - C'| \cong |C - C'| \gg |\varepsilon|$.

There are two cases that may happen. On one hand, $\varepsilon' \gg \varepsilon$ (or $C \gg C'$ meaning that the load

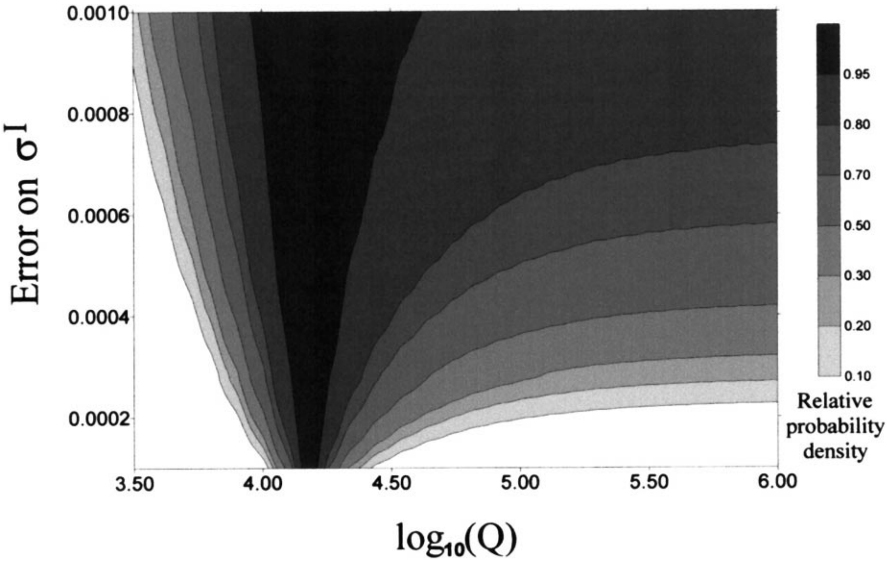

Fig. 6. PDF for $x = \log(Q)$ for increasing error estimates $\Delta\sigma^{\mathrm{I}}_{\mathrm{nd}}$ when positivity is imposed ($\sigma^{\mathrm{I}}_{\mathrm{nd}} > 0$), as well as the null information ($\mu(\log(Q)) = \text{const.}$) in the Bayesian approach.

estimate is much smaller than the true one), the corrected measurement ε' is overestimated and the Q value (which is inversely proportional to ε') is lowered. On the other hand, $(-\varepsilon') \gg \varepsilon$ (or $C \ll C'$), then we have $\varepsilon' < 0$ and hence $Q < 0$. We believe that inaccurate ocean loading leads to a systematic underestimation of the quality factor and can even explain why negative values are sometimes found in standard inversion methods when using gravimetric factors derived from tidal analysis. The problem can be also analysed in terms of probability. Consider two random variables x and y having normal Gaussian PDF:

$$P_x(x) = \frac{1}{\sqrt{2\pi}\,\Delta x}\,\mathrm{e}^{-\frac{(x-\bar{x})^2}{2(\Delta x)^2}},$$

$$P_y(y) = \frac{1}{\sqrt{2\pi}\,\Delta y}\,\mathrm{e}^{-\frac{(y-\bar{y})^2}{2(\Delta y)^2}}, \tag{14}$$

where Δx, Δy are the standard deviations and $\bar{x}$, $\bar{y}$ the mean values of the distributions.

The PDF of the sum $z = x + y$ is:

$$P_z(z) = P_x(x) \otimes P_y(y) = \int P_x(t)\,P_y(z-t)\,\mathrm{d}t$$
$$= \frac{1}{\sqrt{2\pi}\,\Delta z}\,\mathrm{e}^{-\frac{(z-\bar{z})^2}{(\Delta z)^2}}, \tag{15}$$

where $\bar{z} = \bar{x} + \bar{y}$ and $\Delta z = \sqrt{\Delta x^2 + \Delta y^2}$. In our case, we have $x = \varepsilon + C$ (tidal measurement + exact ocean load) and $y = -C'$ (estimated ocean loading correction) leading to $z = \varepsilon' = \varepsilon + C - C'$ (tidal measurement + ocean loading residual).

If there were no loading effect, one would have measured the parameter ε with an error $\Delta\varepsilon \ll \varepsilon$ usually small (see, e.g., Hinderer et al., 1994b, 1995) so that a high Q factor would have been found. But here $\Delta\varepsilon' = \sqrt{(\Delta\varepsilon)^2 + (\Delta C')^2} \cong \Delta C'$ depending on the uncertainty in the estimated ocean load which is usually large (see, e.g., Neuberg et al., 1987, 1990). In terms of probability, the standard deviation of ε' is much larger than the one of ε, so that the corrected value ε' has a relatively strong probability to be large, leading to a strong probability for Q to be small or even negative (see Fig. 7). This shows that ocean loading corrections are crucial in gravity mea-

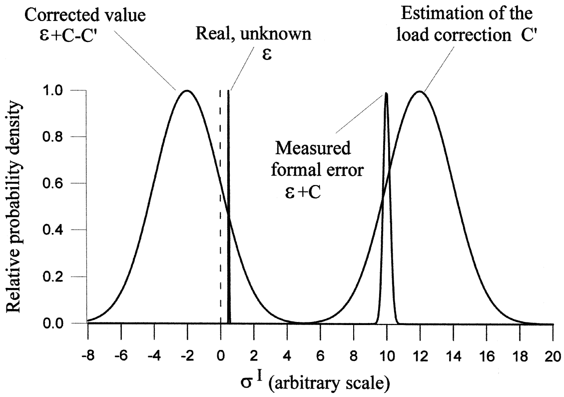

Fig. 7. PDF of $\sigma^{\mathrm{I}}_{\mathrm{nd}}$ after ocean loading corrections. If the estimate of the ocean loading contribution C' is different from the true value C, it leads either to larger estimates of $\sigma^{\mathrm{I}}_{\mathrm{nd}}$ (and hence lower Q values), or to $\sigma^{\mathrm{I}}_{\mathrm{nd}} < 0$ (and hence negative Q values).

surements for each tidal wave which has a significant weight in the inversion. It is therefore highly recommended to use the best available ocean loading corrections as the ones derived from satellite altimetry (see Melchior and Francis, 1996; Llubes and Mazzega, 1997). This is needed before retrieving in the best conditions the FCN resonance parameters, especially Q, from high precision gravity data as the ones which are presently assembled within the frame of the Global Geodynamics Project (GGP) of a worldwide network of superconducting gravimeter (see Crossley and Hinderer, 1995).

5. Conclusion

This paper is devoted to different aspects involved in the determination of the FCN parameters (eigenperiod, damping, strength) from tidal gravimetric data. After a review of different models expressing the resonance process in the tidal gravity changes, we introduce the positivity of the quality factor and propose a new formulation. We then review the different methodologies which are classically used in the retrieval (analytical, linearised least squares, generalised non-linear inverse, stochastic), and propose a new Bayesian approach to the problem which is able to provide the most complete and reliable information on the FCN inversion.

Such a probabilistic approach using tidal gravimetric data is useful in many respects. It first allows us to better understand the systematic underestimate of the quality factor Q which was found in previous studies. We can show that neglecting the null information criterion on the damping leads to a paradox, namely that the most probable Q estimate is moved to lower values according to increasing errors on the imaginary part of the eigenfrequency $\sigma^{\mathrm{I}}_{\mathrm{nd}}$; on the contrary, when introducing the right parameterisation, increasing this error only enlarges the probability law without unreasonably shifting the most probable value. Taking the null information into account also allows us to exclude non-physical negative Q values. Finally, the predicted values become compatible with those derived from space geodesy (VLBI). We could also point out the crucial role played by the accuracy in the ocean loading corrections. It appears clearly that these corrections have to be improved in order to fully exploit tidal gravity data as a precise tool for investigating the FCN resonance.

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